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SCALE COVARIANCE AND G-VARYING COSMOLOGY. II. THERMODYNAMICS, RADIATION, AND THE 3 K BACKGROUND

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ABSTRACT

Within the framework of scale-covariant theory of gravitation, a semiclassical description of particles and photons is given. New thermodynamic relations consistent with the modified conservation equations are derived. Application to a system of radiation shows that the observed 3 K background radiation can be interpreted, within the present framework, as a remnant of *equilibrium radiation* in the past.

As the theory postulates a nonstandard coupling between gravitation and electrodynamics, the assumption that Einstein's theory of gravitation is unchanged forces modifications at the atomic level. The use of Minkowskian spacetime in atomic physics is found to be adequate only over small but not large time scales compared to the age of the universe. As a result, it is shown that the relation between energy ϵ_ν and the frequency ν of a free photon must be $\epsilon_\nu = h\nu/\beta$. Possible observational consequences of this new relation are discussed.

Subject headings: cosmic background radiation — cosmology — gravitation — relativity

I. INTRODUCTION

In a previous paper (Canuto *et al.* 1977, hereafter Paper I), a scale-covariant theory of gravitation was proposed so that a possible variation of the gravitational constant G is understood in terms of the relative variation of gravitational and electrodynamical clocks. Such a variation is provided by a scaling factor $\beta(x)$ which plays the role of an arbitrary gauge function in the proposed theory. Depending on the gauge conditions imposed, the theory can be made consistent with some form of Dirac's Large Numbers Hypothesis, and we suggested that Dirac cosmology should be studied in terms of the scale-covariant theory. Ideally, one would like to have a unified theory of gravitation and all the dynamics of microscopic physics which incorporates naturally a variation of the gravitational constant. Such a complete theory would provide a framework for the description of all physical phenomena. However, the proposed scale-covariant theory is far from being complete. We have merely the framework for studying gravitational phenomena using nongravitational measuring instruments. Since astrophysics involves much more than purely gravitational phenomena, we set out in the present paper to clarify the basis of our astrophysical description. Based on this understanding, we shall investigate the compatability of our theory with cosmological data in subsequent papers in this series (Canuto and Owen 1979; Canuto, Hsieh, and Owen 1979*b*).

Such a clarification is important because what the scale-covariant theory modifies is not Einstein's theory of gravitation itself, but the relation between atomic dynamics and spacetime. Thus we find that in atomic units the conservation laws are modified, and the particle trajectories are not geodesics but in-geodesics. Hence when considering astrophysical phenomena, if any well known relation is invoked, painstaking care should be taken to ensure that such a relation is consistent with the scale-covariant framework. In particular, the relation must be compatible with the modified conservation laws. There exist analyses (for a critical review see Canuto, Hsieh, and Owen 1979*a*) which conclude that known astrophysical phenomena can rule out a varying gravitational constant. But such analyses do not take into account the fact that simply allowing for a varying G necessitates a change in the entire theoretical framework; hence they demonstrate their own logical inconsistency rather than the incompatibility of varying G with observations.

Short of a complete theory, we shall build up models and develop results from the established framework: the Integrable Weyl (IW) Space, which we shall review briefly in the next section. Our description of particles will be based on the in-geodesics of the IW space. In particular, photon momentum, observed energy, and frequency will be defined in § III, in terms of the tangent vectors of the null, in-geodesics. A cosmological redshift relation will also be derived, using conservation laws established in § II. In § IV, we describe the necessary modifications of classical thermodynamics so that the latter is compatible with our modified conservation laws. Thermodynamic relations will be derived for specific systems, and results for classical ideal gas will be singled out for comparison with the standard theory. Radiation as a thermodynamical system will be studied in § V. The classical derivation

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of Wien's displacement law will be repeated here to demonstrate the necessary modifications in the present framework. Finally, the properties of the equilibrium radiation spectrum given by Wien's law will be used to explain the observed background radiation as a remnant of the past.

Many exact results derived below can alternatively be obtained by more intuitive arguments. By displaying some details of the derivation, we wish to emphasize that when considering the consequences of the scale-covariant theory of gravitation, *one must ascertain that all equations used, no matter how obvious they are in the corresponding standard theory, must be either consequences of equations stipulated in the formulation of the new theory or compatible, independent assumptions.*

II. MATHEMATICAL PRELIMINARIES

In this section, we shall state some of the basic assumptions of the theory and give a brief review of its mathematical formulation. Some new results which are useful for subsequent discussions will then be derived. In particular, Killing vectors will be introduced to derive conservation laws along in-geodesics. These laws will be used on the one hand to derive the cosmological redshift relations for photons and on the other hand as a basis for discussions of observed single particle energy. The latter will then be used to give a kinetic interpretation of the thermodynamic results in §§ IV and V.

a) Review of Co-tensors

The fundamental assumptions we have made are: (1) There exist two dynamical units of length: gravitational and electrodynamical. (We shall also call these, respectively, Einstein and atomic units.) (2) Gravitational phenomena are correctly described by Einstein's general theory of relativity which uses implicitly a gravitational unit of length. In standard theory, it is further assumed that the two units are *constant* multiples of each other. We relax this and assume instead: (3) If $d\bar{s}$, ds denote the line element of spacetime measured respectively in gravitational and atomic units, then

$$d\bar{s} = \beta(x)ds\,. \tag{2.1}$$

It follows then that the geometry of spacetime as measured in the two units are conformal transformations of each other:

$$\bar{g}_{\mu\nu}(x) = \beta^2(x)g_{\mu\nu}(x)\,. \tag{2.2}$$

Since the Einsteinian spacetime described by $\bar{g}_{\mu\nu}$ is Riemannian, the atomic spacetime is an IW space. (For details, see Paper I.) We note that in our formulation $\beta(x)$ or, more precisely, the derivatives of $\beta(x)$ are measurable. An example was given in Paper I, where such a measurement was linked with the variation of the periods (in atomic units) of planetary orbits. We shall give more examples below by which $\beta_{,\mu}$ can in principle be measured. We remark also that $\beta(x)$ is not a dynamical variable in the scale-covariant theory of gravitation. While we do suggest its observational determination, we can also treat it theoretically as a gauge function which is to be determined by the imposition of gauge conditions. (Our choice of the word *gauge* is perhaps unfortunate because its usage in modern physics implies physical irrelevance. We wish to emphasize its meaning, as introduced by Eddington 1924, which connotes a system of units.) For example, we can make use of Dirac's Large Numbers Hypothesis, which makes definite physical statements, to determine the gauge function. In this connection, we also point out that in the present work, we associate gauge conditions only with cosmological considerations and hence the scales of variation of β are also cosmological.

The proper geometrical objects in IW space are the co-tensors which, in addition to the well known tensor properties under coordinate transformation, also transform as

$$A \rightarrow \bar{A} = \beta^{\Pi}A \tag{2.3}$$

under the conformal transformation

$$g_{\mu\nu} \rightarrow \bar{g}_{\mu\nu} = \beta^2 g_{\mu\nu}\,.$$

Here the A represent general co-tensors with indices suppressed, and the exponent Π in (2.3) is called the power of A. Clearly the power of a co-tensor product is the sum of the powers of its factors.

In order to preserve the transformation property (2.3) for the derivatives of co-tensors, the standard covariant differentiation must be replaced by co-covariant differentiation which are defined as follows, for scalars and tensors.

$$S_{*\mu} = S_{;\mu} + \Pi(\ln\beta)_\mu S\,, \tag{2.4a}$$

$$V^\mu{}_{*\nu} = V^\mu{}_{;\nu} + g^\mu{}_\nu(\ln\beta)_\lambda V^\lambda + (1+\Pi)V^\mu(\ln\beta)_\nu - V_\nu(\ln\beta)^\mu\,, \tag{2.4b}$$

$$V_{\mu*\nu} = V_{\mu;\nu} + g_{\mu\nu}(\ln\beta)_\lambda V^\lambda + (\Pi - 1)V_\mu(\ln\beta)_\nu - V_\nu(\ln\beta)_\mu\,. \tag{2.4c}$$

Here an asterisk denotes co-covariant differentiation and a semicolon indicates covariant differentiation, as usual.

Π is the power of the co-tensor to be differentiated. We leave out the comma for partial differentiations of $\ln \beta$.

Generalization to higher rank tensors is trivial, and we give two more examples:

$$A^{\mu\nu}{}_{*\lambda} = A^{\mu\nu}{}_{;\lambda} + g_\lambda{}^\mu(\ln\beta)_\rho A^{\rho\nu} + g_\lambda{}^\nu(\ln\beta)_\rho A^{\mu\rho} + (2+\Pi)(\ln\beta)_\lambda A^{\mu\nu} - A_\lambda{}^\nu(\ln\beta)^\mu - A^\mu{}_\lambda(\ln\beta)^\nu\,, \tag{2.4d}$$

$$A_{\mu\nu*\lambda} = A_{\mu\nu;\lambda} + g_{\mu\nu}(\ln\beta)^\rho A_{\rho\nu} + g_{\nu\lambda}(\ln\beta)^\rho A_{\mu\rho} + (\Pi-2)(\ln\beta)_\lambda A_{\mu\nu} - A_{\lambda\nu}(\ln\beta)_\mu - A_{\mu\lambda}(\ln\beta)_\nu\,. \tag{2.4e}$$

Since we have stipulated that general relativity (GR) correctly describes gravitational phenomena, the GR equations are considered valid in gravitational units. To express these equations in atomic units, one needs only make a conformal transformation. The result is always the replacement of covariant differentiations by co-covariant differentiations. For example, the geodesic equation in gravitational units is valid:

$$\bar{v}^\mu{}_{;\nu}\bar{v}^\nu = 0\,. \tag{2.5a}$$

Expressed in atomic units, we can simply write

$$v^\mu{}_{*\nu}v^\nu = 0\,. \tag{2.5b}$$

Similarly,

$$\bar{T}^{\mu\nu}{}_{;\nu} = 0 \tag{2.6a}$$

is valid by virtue of Einstein's field equations, which we accept. In atomic units, we have

$$T^{\mu\nu}{}_{*\nu} = 0\,. \tag{2.6b}$$

Equations (2.5b) and (2.6b) remain merely symbolic until the co-tensor powers of v^μ and $T^{\mu\nu}$ are specified. We shall refer to Paper I for the details of such specification process. Briefly, we note that there are two main types of co-tensors: (1) Geometrical co-tensors which are constructed from the basic quantities of IW space: $g_{\mu\nu}$, dx^μ, and β whose co-tensor powers are by definition 2, 0, and -1, respectively. As an example, $ds = (g_{\mu\nu}dx^\mu dv^\nu)^{1/2}$ is easily seen to be of power $+1$. Consequently $v^\mu \equiv dx^\mu/ds$ is of power -1. (2) Physical quantities such as energy-momentum density are usually represented by co-tensors whose powers are gauge dependent. Consequently, the way atomic mass or energy converts into a length is a gauge-dependent relation. (In fact the main motivation of the scale-covariant formulation is to allow for such a freedom and at the same time maintain consistency among the equations.) We need only point out here that for the kinds of gauge conditions we have imposed so far, the co-tensor power of physical quantities are parametrized by Π_g, the co-scalar power of the gravitational constant. As an illustration, we quote from Paper I:

$$\Pi(GT_{\mu\nu}) = 0\,,$$

since it is equal to a geometrical tensor having zero power. Consequently,

$$\Pi(T_{\mu\nu}) = -\Pi_g \tag{2.7}$$

and

$$\Pi(T^{\mu\nu}) = \Pi(g^{\mu\lambda}g^{\nu\rho}T_{\lambda\rho}) = -4 - \Pi_g\,. \tag{2.8}$$

Furthermore, as we may write

$$T_{\mu\nu} = \rho v_\mu v_\nu\,;$$

thus,

$$\Pi(\rho) + 2\Pi(v_\mu) = \Pi(\rho) + 2\Pi(g_{\mu\nu}v^\nu) = -\Pi_g\,,$$

$$\Pi(\rho) = -2 - \Pi_g\,. \tag{2.9}$$

As ρ is an energy density, using the convention that the velocity of light c is dimensionless (for time interval is treated as a length), we find from (2.9) that the power of mass is given by

$$\Pi_m \equiv \Pi(m) = 1 - \Pi_g\,. \tag{2.10}$$

Thus with Π_m expressed in terms of Π_g, length and time having power $+1$, the power of any physical quantity is easily obtained from the dimensions of the quantity. For example, the power of number density would be -3, and the power for mass (or energy) density is simply $\Pi_m - 3 = -2 - \Pi_g$, agreeing with (2.9) and (2.10). The only exception to this rule is β, which is dimensionless and has power -1. The reason for this exception is clear: *It is not a physically measurable quantity within one system of units.*

b) *Killing Vectors*

To facilitate the discussion of conservation laws below, we introduce here the notion of IW Killing vectors. In Riemannian space, if

$$x^\mu \to x'^\mu = x^\mu + \epsilon\bar{\xi}^\mu \tag{2.11}$$

is an infinitesimal coordinate transformation under which the metric $\bar{g}_{\mu\nu}$ is invariant, then

$$\bar{\xi}_\mu = \bar{g}_{\mu\nu}\bar{\xi}^\nu \tag{2.12}$$

is a Killing vector and satisfies the equation

$$\bar{\xi}_{\mu;\nu} + \bar{\xi}_{\nu;\mu} = 0\,. \tag{2.13}$$

From (2.11), $\bar{\xi}^\mu$ plays the role of a coordinate function and hence $\Pi(\bar{\xi}^\mu) = 0$; $\Pi(\bar{\xi}_\mu) = 2$.

The metric of the corresponding IW space is given by

$$g_{\mu\nu} = \beta^{-2}\bar{g}_{\mu\nu}\,, \tag{2.14}$$

and

$$\xi_\mu = \beta^{-2}\bar{\xi}_\mu\,; \qquad \xi^\mu = \bar{\xi}^\mu\,, \tag{2.15}$$

by virtue of the definition of transformation, so that it can be trivially shown that the IW Killing vector ξ_μ satisfies

$$\xi_{\mu*\nu} + \xi_{\nu*\mu} = 0\,. \tag{2.16}$$

To avoid confusion, we point out at the risk of belaboring the obvious, that given an IW Killing vector, the metric $g_{\mu\nu}$ is not invariant under the infinitesimal transformation

$$x^\mu \rightarrow x'^\mu = x^\mu + \epsilon\xi^\mu(x)\,.$$

It is the corresponding Riemannian metric

$$\bar{g}_{\mu\nu} = \beta^2 g_{\mu\nu}$$

that is invariant. The reason why such IW Killing vectors play a role in our study of conservation laws stems from a basic assumption in our formulations of the theory: energy is strictly conserved in Einstein units. (This is a consequence of assumption (2) given in the previous subsection; for the Bianchi identities require a conserved energy momentum tensor.)

For illustrative purposes, we shall display explicitly the solution of (2.16) for two specific metrics, chosen for their simplicity and subsequent applications. First we make use of (2.4c) to rewrite equation (2.16) as

$$\xi_{\mu,\nu} + \xi_{\nu,\mu} - \Gamma^\lambda{}_{\mu\nu}\xi_\lambda + 2g_{\mu\nu}(\ln\beta)_\lambda\xi^\lambda = 0 \tag{2.17}$$

We include also solution to the in-geodesic equation (2.5b) which, with the aid of (2.4b), can be written as

$$v^\mu{}_{,\nu}v^\nu + \Gamma^\mu{}_{\nu\lambda}v^\nu v^\lambda + v^\mu(\ln\beta)_\nu v^\nu - (v^\nu v_\nu)(\ln\beta)^\mu = 0\,. \tag{2.18}$$

Without further algebraic details, we summarize that

i) For flat spacetime in Einstein units, the corresponding line element in atomic units is written as

$$ds^2 = \beta^{-2}\eta_{\mu\nu}dx^\mu dx^\nu\,. \tag{2.19}$$

Then

$$\xi_\mu = \beta^{-2}(\omega_{\mu\nu}x^\nu + a_\mu)\,, \tag{2.20}$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$ and a_μ are constants.

If $v_\mu v^\mu = 1$, (2.18) is satisfied by

$$v^\mu = \beta(\gamma, \gamma V^i)\,, \tag{2.21}$$

where V^i are constants and $\gamma = (1 - V^2)^{-1/2}$. There exists also null vector solution for (2.18). We denote such null vector by k_μ, with $k^\mu k_\mu = 0$:

$$k^\mu = \beta\nu_0(1, l^i)\,, \tag{2.22}$$

where the l^i's are constants with

$$\sum_{i=1}^{3} l^i l^i = 1\,,$$

and ν_0 is an arbitrary constant.

ii) For a Robertson-Walker metric

$$ds^2 = dt^2 - R^2\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{2.23}$$

the solutions are

$$\xi_\mu = \left(0, \frac{R^2}{(1-kr^2)^{1/2}}\cos\theta, -R^2 r(1-kr^2)^{1/2}\sin\theta, 0\right), \tag{2.24}$$

$$v^\mu = \left[\left(1+\frac{v_0{}^2}{\beta^2 R^2}\right)^{1/2}, \frac{v_0}{\beta R^2}(1-kr^2)^{1/2}, 0, 0\right], \tag{2.25}$$

$$k^\mu = \frac{\nu_0}{\beta R}\left(1, \frac{(1-kr^2)^{1/2}}{R}, 0, 0\right). \tag{2.26}$$

Note that we have not given the most general solution to equations (2.17) and (2.18). For example, (2.25) and (2.26) give only radial motions. But these solutions will be sufficient as illustrative examples below.

c) *Conservation Laws along In-Geodesics*

We denote the path parameter of in-geodesics by λ. The tangent vector will generally be written as $dx^\mu/d\lambda$. When specializing to particle paths, we use the path length s as path parameter and write the tangent vector as $v^\mu = dx^\mu/ds$.

Given any co-tensor A, we stipulate that the *observed* changes of A along the path are given, as in the standard theory, by

$$\frac{DA}{D\lambda} \equiv A_{;\nu}\frac{dx^\nu}{d\lambda}, \tag{2.27a}$$

where again the indices of A are suppressed. It is convenient to introduce the notation

$$\frac{D_* A}{D_* \lambda} \equiv A_{*\nu}\frac{dx^\nu}{d\lambda}. \tag{2.27b}$$

We observe that if A is a scalar, (2.4a) gives

$$\frac{D_* S}{D_* \lambda} = \frac{DS}{D\lambda} + \Pi(\ln\beta)_\nu \frac{dx^\nu}{d\lambda} S. \tag{2.28}$$

With these preliminaries, we can proceed to derive the conservation laws.

Let ξ_μ and v_μ satisfy (2.16) and (2.5b), respectively. Then by definition (2.27b),

$$\frac{D_*}{D_* s}(\xi_\mu v^\mu) = (\xi_\mu v^\mu)_{*\nu} v^\nu = \xi_{\mu*\nu} v^\mu v^\nu + \xi_\mu v^\mu{}_{*\nu} v^\nu = 0, \tag{2.29}$$

where, in the last step, (2.16) and (2.5b) have been used. On the other hand, since $\Pi(\xi_\mu u^\mu) = \Pi(\xi_\mu) + \Pi(u^\mu) = 1$, combining (2.29) with (2.28), we find

$$\frac{D}{Ds}(\beta\xi_\mu v^\mu) = 0, \quad \text{or} \quad \beta\xi_\mu v^\mu = \text{constant} \tag{2.30}$$

along the integral path of v^μ. Analogously, given a null path with tangent vector denoted by $k^\mu = dx^\mu/d\lambda$, we get

$$\frac{D}{D\lambda}(\beta\xi_\mu k^\mu) = 0, \quad \text{or} \quad \beta\xi_\mu k^\mu = \text{constant} \tag{2.31}$$

along the integral path of k^μ. It can be easily ascertained that the sets (2.20)–(2.22) and (2.24)–(2.26) are consistent with the relations (2.30) and (3.31). These relations will be useful when we consider the observed energy of individual particles in the next section.

We note that for a metric of the type (2.23) where only spatial symmetry is assumed, the Killing vector is always spacelike, and hence $\xi_\mu v^\mu$ picks out the spatial components of v. Considering this from a different point of view, let $u^\mu = (1, \mathbf{0})$ be the comoving velocity for (2.23). The velocity u^μ is clearly geodesic, for it can be obtained from (2.25) by setting $v_0 = 0$. It is easy to verify that

$$\xi_\mu v^\mu = \cos\theta \cdot [(u_\mu v^\mu)^2 - 1]^{1/2}.$$

Along the radial motion of v^μ, $\cos\theta$ is constant, hence we have as a special case of (2.30)

$$\frac{D}{Ds}[\beta((u_\mu v^\mu)^2 - 1)^{1/2}] = 0. \tag{2.32a}$$

In fact, one can write more generally,

$$\frac{D}{D\lambda}\left[\beta\left(\left(u_\mu \frac{dx^\mu}{d\lambda}\right)^2 - \epsilon\right)^{1/2}\right] = 0 \tag{2.32b}$$

where $\epsilon = (dx^\mu/d\lambda)dx_\mu/d\lambda$ vanishes for null paths. In the form (2.32), it can be seen that it is the spatial part of v^μ, or the momentum, that is conserved in spacetime with only spatial symmetry.

There is yet another simple conservation law we can include in the discussion. Suppose the energy momentum tensor is written as

$$T^{\mu\nu} = \rho v^\mu v^\nu \,. \tag{2.33}$$

By virtue of (2.6b) and (2.5b), it is easy to show that

$$(\rho v^\mu)_{*\mu} = 0 \,. \tag{2.34}$$

From (2.9), we find

$$\Pi(\rho v^\mu) = -3 - \Pi_g\,;$$

hence, with (2.4b) we obtain

$$(\rho v^\mu)_{;\mu} + (1 - \Pi_g)\rho(\ln\beta)_\mu v^\mu = 0\,, \tag{2.35a}$$

which can be written as

$$\beta^{1-\Pi_g}\rho V = \text{constant}\,. \tag{2.35b}$$

Here V is a volume element comoving with v^μ. When v^μ is identified with u^μ, (2.35b) is simply the statement of modified mass conservation law as described in Paper I. For future reference, we note that if the radiation energy momentum tensor is written as

$$T_\gamma{}^{\mu\nu} = Ek^\mu k^\nu\,, \tag{2.36}$$

a derivation analogous to the above gives

$$(Ek^\mu)_{;\mu} + (1 - \Pi_g)E(\ln\beta)_\nu k^\nu = 0\,, \tag{2.37}$$

whose physical interpretation will be given in the next section.

III. DESCRIPTION OF PHOTONS

a) Photon Frequency and Cosmological Redshift

Thus far, we have at our disposal geometrical paths for macroscopic bodies, as they are the object of investigation in a classical theory of gravitation. By the weak equivalence principle, we can stipulate that individual elementary particles also move along the same in-geodesic paths, provided geometric measurements are made in atomic units. In this section, we shall further define the physically measurable quantities of particles in terms of the tangent vectors of the in-geodesics. For massive particles the procedure is identical to that of a semiclassical description of particles: The 4-velocity $v^\mu \equiv dx^\mu/ds$ is the tangent vector of an in-geodesic satisfying (2.18) and having unit normalization. If the particle has a mass m, the 4-momentum is defined as

$$p^\mu = mv^\mu\,. \tag{3.1}$$

For an observer having 4-velocity u^μ, the measured energy of the above particle is

$$\epsilon_p = u_\mu p^\mu\,. \tag{3.2}$$

For photons, we start with the null in-geodesic whose tangent vector $k^\mu \equiv dx^\mu/d\lambda$ by definition has vanishing norm:

$$k^\mu k_\mu = 0\,. \tag{3.3}$$

Furthermore, to promote the symmetry between descriptions of photon and massive particles, we stipulate that $\Pi(k^\mu) = -1$, so that k^μ also satisfies (2.18) (with the last term on the left-hand side of [2.18] vanishing by virtue of [3.3]). Analogous to (3.1), we define the photon 4-momentum by

$$p^\mu = hk^\mu\,, \tag{3.4}$$

where h is a constant. Note that for a null path, the differential of path parameter $d\lambda$ is not uniquely defined as is ds for particles. Thus, the same null path can give rise to different photon momenta. Equation (3.2) will be applied to both particles and photons to obtain the observed energy.

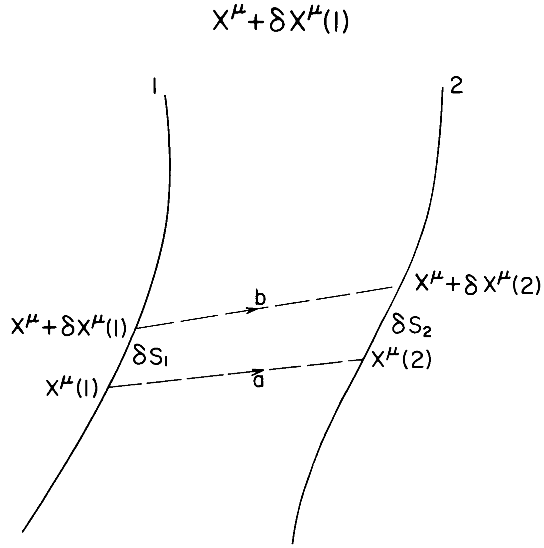

FIG. 1.—Definition of photon frequency from propagation of successive signals

To complete our description of photons, we must relate the frequency of a photon to the tangent vector k^μ. However, it is not our intention here to give a wave description of photons as well. Frequencies are *defined* in terms of the inverse of the duration between successive pulses of radiation, such as that given by Schrödinger(1956) in his discussion of null geodesics in standard relativity. Each pulse travels along a null in-geodesic so that, in Figure 1, lines 1 and 2 are respectively source and observer paths whereas lines *a* and *b* are photon paths. Thus the ratio of the frequencies is given by

$$\nu_1/\nu_2 = \delta s_2/\delta s_1 \, . \tag{3.5}$$

To express the above ratio in terms of the tangent vector k^μ, we follow Schrödinger's approach and introduce first a variational principle which gives rise to equation (2.18). Several versions of such a variational principle have been given (Dirac 1973; Bouvier and Maeder 1978), but they apply only for tangent vectors having nonvanishing norm. The modified version we give below remedies this defect and is therefore applicable to the case of current interest. Toward this end, we introduce a scale-invariant path parameter τ so that

$$I = \int d\tau \beta^2 g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \tag{3.6}$$

is explicitly invariant under conformal transformation. The affine parameter $d\lambda$ is related to $d\tau$ by

$$d\lambda = \beta^{-1} d\tau \, . \tag{3.7}$$

The Euler-Lagrange equation from the variational principle

$$\delta I = 0 \tag{3.8}$$

is given by

$$\frac{d}{d\tau}\left(\beta^2 g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) - \tfrac{1}{2}(\beta^2 g_{\nu\rho})_\mu \frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0 \, . \tag{3.9}$$

With the aid of (3.7), it is easy to show that (3.9) can be rewritten in the form of (2.18); and in particular if

$$\frac{dx^\mu}{d\lambda}\frac{dx_\mu}{d\lambda} = 0 \, ,$$

equation (3.9) reduces to

$$k^\mu{}_{*\nu} k^\nu = k^\mu{}_{;\nu} k^\nu + k^\mu (\ln \beta)_\nu k^\nu = 0 \, . \tag{3.10}$$

Returning to the computation of the ratio (3.5), we consider again the variation (3.8), this time including the variation at the endpoints:

$$0 = \delta \int d\tau \beta^2 g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 2\left[\beta^2 g_{\mu\nu}\frac{dx^\mu}{d\tau}\delta x^\mu\right]_1^2 .$$

The total variation is zero because the integrand vanishes along both paths a and b of Figure 1. The second equality of the above equation follows as a consequence of (3.9). We have therefore

$$\beta^2(1)g_{\mu\nu}(1)\frac{dx^\mu}{d\tau}(1)\delta x_1{}^\nu = \beta^2(2)g_{\mu\nu}(2)\frac{dx^\mu}{d\tau}(2)\delta x_2{}^\nu .$$

Thus,

$$\frac{\delta s_2}{\delta s_1} = \left(\beta^2 g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{\delta x^\nu}{\delta s}\right)_1 \Big/ \left(\beta^2 g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{\delta x^\nu}{\delta s}\right)_2 = \frac{(\beta k^\mu u_\mu)_1}{(\beta k^\mu u_\mu)_2}, \tag{3.11}$$

where we have written k^μ for $dx^\mu/d\lambda$ and u^μ for $\delta x^\mu/\delta s$. Equations (3.5) and (3.11) combine to give

$$\nu_1/\nu_2 = (\beta u_\mu k^\mu)_1/(\beta u_\mu k^\mu)_2 . \tag{3.12}$$

Thus, we can stipulate that given a null vector k^μ, the observed photon frequency is given by

$$\nu = \beta u_\mu k^\mu . \tag{3.13}$$

For β identically equal to unity, the above analysis reduces to that given by Schrödinger (1956), and equations (3.12) and (3.13) reduce to those of standard relativity. It should be noted that β appearing in (3.13) is a function of spacetime, whose value associated with each measurement of the frequency depends only on the spacetime point at which the measurement is made and does not depend on the history of the photon being measured.

With equation (3.12), the cosmological redshift relation can be easily derived for a Robertson-Walker metric, using equations (2.25) and (2.26). In particular, we set $v_0 = 0$ in (2.25) so that the resulting velocity field

$$u^\mu = (1, \mathbf{0})$$

represent comoving observers. Computing the product $\beta u_\mu k^\mu$, we find

$$\nu_1/\nu_2 = R_2/R_1 . \tag{3.14}$$

Equation (3.14) is seen to be identical to the redshift relation of standard cosmology, and indeed can be derived directly from the form of the metric given by (2.23). The primary objective of the above analysis has been the clarification of our model of photons and the introduction of equation (3.13).

b) Photon Energy

Among the consequences of (3.13), we note that, together with (3.2), we find for the observed photon energy:

$$\epsilon_\gamma = u_\mu p^\mu = h u_\mu k^\mu = \frac{h}{\beta}\nu . \tag{3.15}$$

Thus the ratio of the observed energy and frequency of a photon is not a constant, but a function of spacetime. We hasten to stress that we are not asserting that Planck's constant actually varies in such a way that relative atomic energy levels would change with time. For it has been stipulated that an atomic unit of length exists in the sense that in the limit where gravitation is irrelevant to atomic dynamics, standard atomic physics holds. Since the atomic unit of length l_0 is derived from the atomic mass, velocity of light, and Planck's constant, the existence of l_0 and the assumption of constant atomic mass and velocity of light implies the constancy of Planck's constant as it appears in atomic dynamics. These stipulations are of course compatible with the constancy of the fine structure constant (see, e.g., Bahcall and Schmidt 1967; Wolfe, Brown, and Roberts 1976). Furthermore, since (3.15) states that ϵ_γ/ν is a function of spacetime, independent of the history of photons, the experiments of Solheim, Barnes, and Smith (1976) and Baum and Floretin-Nielson (1976) are irrelevant as a test of (3.15). This is because, even though a photon was emitted in the past, as long as its energy and frequency are measured at present, their ratio would not be different, according to (3.15), from that of a laboratory-produced photon. To test the relation, both energy and frequency must be measured at two spacetime points. Stated differently, such experiments can be considered measurements of the derivatives of β.

Having argued for the viability of equation (3.15), we next examine some alternatives and show that they lead to less desirable consequences. A more thorough discussion of the physical significance of (3.15) will then follow. We recall again that with (3.2), (3.4), and (3.13), we are *defining* the observable quantities of a photon in terms of

k^μ, the tangent vector pertaining to the geometrical model of the photon. Thus, in principle we can change one or more of the above three defining equations.

Equation (3.2) defining the observed energy in terms of the 4-momentum works very well for massive particles. Since, given a 4-momentum, one should not have to specify its normalization before computing its observed energy, we insist that (3.2) applies also to photons.

In place of (3.13), we could simply define the photon frequency as

$$\nu = u_\mu k^\mu , \tag{3.13$'$}$$

so that the usual relation

$$\epsilon_\gamma = h\nu \tag{3.15$'$}$$

is obtained. The difficulty with (3.13′) is that, one could then show with (2.25) and (2.26) that

$$\nu_1/\nu_2 = (\beta R)_2/(\beta R)_1 \tag{3.14$'$}$$

for the Robertson-Walker metric (2.23). And it can be seen that relation (3.14′) is not compatible with (3.5) which conforms to our intuitive notion of observed frequency, and which we hold to be valid independent of formulations.

Finally we note the possibility of defining

$$k'^\mu \equiv \beta k^\mu , \qquad \nu \equiv u_\mu k'^\mu , \quad \text{and} \quad p^\mu \equiv hk'^\mu . \tag{3.16}$$

It can be easily seen that with (3.16), (3.14) and (3.15′) are simultaneously valid. We have rejected this set of definitions because it leads to discrepancy between the observed photon and particle energy. This is easily illustrated when we consider a spacetime given by the metric (2.19). From (2.22) and (3.16), we have

$$k'^\mu = \beta^2\nu_0(1, l^i) . \tag{3.17}$$

Setting $V^i = 0$ in (2.21), we have the comoving in-geodesic

$$u^\mu = \beta(1, \mathbf{0}) , \tag{3.18}$$

which we assume to represent the observer velocity. The more general form of (2.21) gives the particle 4-velocity

$$v^\mu = \beta(\gamma, \gamma V^i) . \tag{3.19}$$

Thus we find

$$\epsilon_{\text{particle}} = u_\mu(mv^\mu) = \gamma m , \tag{3.20a}$$

$$\epsilon_{\text{photon}} = u_\mu(hk'^\mu) = \beta h\nu_0 , \tag{3.20b}$$

which asserts that in a spacetime having conserved particle energy, the photon energy is not conserved. We find this conclusion unacceptable and have therefore chosen the description of photons as given at the beginning of this section.

The above discussion can be stated in more general terms: Equations (2.30) and (2.31) give the conserved quantities along in-geodesics. If a timelike IW Killing vector ξ_μ exists, the conserved quantities are energies. In this case, it can be shown that

$$u_\mu = \beta\xi_\mu ,$$

where u_μ is the 4-velocity of the comoving observer. Consequently, it can be seen from (2.31) that we must define the photon momentum by (3.4) and not (3.16), so that (2.31) gives the conserved photon energy.

We recall that equation (3.2) is a formal definition of particle energy whose unit of reference is given, for example, by an elementary particle mass such as that of the electron. It must be stressed that this definition cannot be applied to measurements of internal energies of atomically bound systems. To give an expression for the atomic interaction energy lies beyond the intended scope of the present paper.

At this point, one may well ask about the advisability of modifying a well known relation, as we have done so by proposing equation (3.15). To appreciate the physical significance of this equation, it is important to note that by speculating a nonstandard coupling between atomic and gravitational dynamics, and furthermore, by stipulating that Einstein's theory of gravitation is unmodified, we are necessarily forcing modifications on the atomic level. In particular, from our basic assumption (2.2), it can be seen that a *gravitational flat region of spacetime*, i.e., $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$, *would appear curved in atomic units due to the nonconstancy of* β. This is fundamentally different from the conventional assumption of Minkowskian spacetime in atomic physics. For this reason, one should not be surprised that the ratio of photon energy to frequency can vary as indicated by (3.15). One possible interpretation is that along with the postulate of a varying G, one must simultaneously admit a cosmological input into local atomic physics. The scaling function β can therefore be viewed as a cosmic field which interacts with atomic dynamics, and equation (3.15) can be considered as the manifestation of such an interaction. We mention also that we envision a complete theory of unified electrodynamics and gravitation in which β plays a role similar to that of the Higgs field in the Weinberg-Salam theory of weak and electromagnetic interaction (Canuto, Hsieh, and Adams 1977). From this viewpoint, modification of atomic dynamics is clearly expected.

For the purposes of the present paper, since a complete, unified theory is yet to be constructed, β will be considered as a cosmic field given *a priori*. As discussed earlier, for sufficiently slowly varying β, measurements of the effects suggested in (3.15) has yet to be devised. Nevertheless, we believe that the theoretical investigations presented here should assist us in obtaining clues as to how a complete theory can be constructed consistent with the assumptions of scale-covariant gravitation as described in § II*a*. Toward that goal, it is also useful to note that in spite of the appearance of the modified equations, we do recover some expected physical results in the gravitation free limit.

As can be seen from (2.2), the metric in atomic units in this limit is given precisely by (2.19). We have already shown earlier in this section that the photon energy is constant for such a metric. This result can be seen more directly from (3.15). We note that in (2.19), β^{-1} plays the role of the scale factor in a Robertson-Walker metric. Hence from the cosmological redshift relation (3.14), we find by replacing R with β^{-1},

$$\nu\beta^{-1} = \text{constant}\,,$$

which in turn gives constant photon energy according to (3.15), as conventionally expected.

A further example is given by the motion of a force-free particle: One may suspect that since the equation of motion is changed from geodesic to in-geodesic equation, the usual constant velocity motion cannot be recovered in the absence of gravitation. However, we note again that the absence of gravitation is characterized by the metric (2.19) for which we found the 4-velocity solution v^μ of the in-geodesic equation to be given by (2.21). From this form of v^μ and the metric, it is easy to see that the *observed* velocity is given by the constants V^i. Here again, the expected physical results is recovered provided the *correct* gravitation free limit is taken.

c) Adiabatic Photon Conservation

We can now proceed to give the physical interpretation of equation (2.37). A beam of photons moving along in-geodesics has energy-momentum tensor given by (2.36). A local observer with 4-velocity u_μ will find a radiation energy density

$$\rho_\gamma = u_\mu u_\nu T_\gamma{}^{\mu\nu} = E(u_\mu k^\mu)^2\,, \tag{3.21}$$

which can be written as

$$\rho_\gamma = n_\gamma \epsilon_\gamma = \Sigma \frac{\nu}{c} \epsilon_\gamma\,, \tag{3.22}$$

where n_γ is the photon number density and ϵ_γ is the energy of each individual photon. We have also expressed n_γ in terms of the surface density Σ and the frequency of the photons. From the above two equations and equation (3.2), (3.4), (3.13), we find

$$E = h\Sigma\beta\,, \tag{3.23}$$

where we have again put the velocity of light $c = 1$. Furthermore, it can be shown that if A is the cross sectional area of the beam,

$$k^\mu{}_{:\mu} = \frac{d}{d\lambda} \ln (A\beta^{-1})\,. \tag{3.24}$$

Combining (3.23) and (3.24) with (2.37), we find

$$\frac{d}{d\lambda}(\Sigma A\beta^{1-\Pi_g}) = 0 \qquad \text{or} \qquad \Sigma A\beta^{1-\Pi_g} = \text{constant} \tag{3.25}$$

along the beam. ΣA is the total number of photons going through the cross section of the beam. If we set $\beta = 1$, we recover the equations of standard relativity, and equation (3.25) states simply that the photon number is conserved. More generally, in the scale-covariant framework, (3.25) states that the number of photons along the beam of radiation may vary. Just as we allow for spontaneous creation of matter in our formulation, as indicated by equation (2.35b), equation (3.25) allows for the spontaneous creation of photons. It should be noted that if we choose the gauge with no spontaneous matter creation, $\Pi_g = 1$ (see Paper I), photon number would also be conserved along the beam. This is a further indication that we have given a consistent description of photons, for one should not be able to independently stipulate the spontaneous creation of particles of different masses.

IV. MODIFIED THERMODYNAMICS

We consider here thermodynamics not as a limit of statistical mechanics, but as a classical theory consisting of deductions from a set of postulates which are consistent with observed phenomena. It is well known that the first law of thermodynamics is a way of stating the conservation of energy. Clearly with the conservation law modified, the standard thermodynamics cannot be retained without modification.

a) General Formulation

We start by examining the energy conservation law which can be obtained by multiplying (2.6b) with u_μ. Using (2.4d), (2.8) and noting that $T^{\mu\nu}$ is symmetric, we find

$$u_\mu(T^{\mu\nu}{}_{;\nu} + (2 - \Pi_g)(\ln\beta)_\nu T^{\mu\nu} - (\ln\beta)^\mu T^\nu{}_\nu) = 0\,. \tag{4.1}$$

Equation (4.1) holds for general $T^{\mu\nu}$. To better illustrate the difference between (4.1) and the corresponding equation in the standard theory, we consider the simple case for which $T^{\mu\nu}$ has the ideal fluid form

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - pg^{\mu\nu}\,. \tag{4.2}$$

Equation (4.1) becomes

$$\dot\rho + (\rho + p)\theta + [(1 - \Pi_g)\rho + 3p]d(\ln\beta)/dt = 0\,, \tag{4.3}$$

where

$$\theta \equiv u^\mu{}_{;\mu} = \dot V/V \tag{4.4}$$

and V is the comoving volume. By simple algebraic manipulation, assuming homogeneity over V, we obtain an equation of the form familiar in classical thermodynamics:

$$dU + pdV + [(1 - \Pi_g)\rho + 3p]Vd\phi = 0\,, \tag{4.5}$$

where

$$\phi \equiv \ln\beta\,; \qquad U = \rho V\,. \tag{4.6}$$

Comparison with the classical adiabatic conservation law

$$dU + \sum X_i dx_i = 0$$

suggests that we can treat ϕ in (4.5) formally as an additional thermodynamic parameter. This means that for a system within an adiabatic partition, such as the ones postulated in standard thermodynamics, any change of state of the system due to a variation of ϕ is to be interpreted as a change of internal energy. Thus, when the system is not in an adiabatic enclosure, we have more generally

$$\delta Q = dU + pdV + [(1 - \Pi_g)\rho + 3p]Vd\phi\,, \tag{4.7}$$

which we shall take as the statement of modified first law of thermodynamics.

We need not alter the second law. Among many equivalent forms we shall take, for its logical simplicity, Caratheodory's principle as the statement of the second law. It asserts that near any given equilibrium state, there exist adiabatically inaccessible states. It can then be shown that there exists an integration factor for δQ so that

$$\delta Q = \Theta dS = dU + pdV + [(1 - \Pi_g)\rho + 3p]Vd\phi\,. \tag{4.8}$$

Equation (4.8) will be the basis of our analysis of the thermodynamic behavior of various systems. Before proceeding further, we shall make some additional remarks concerning the above modification of classical thermodynamics.

In standard theory of thermo-hydrodynamics, one often separate out the viscous part of the energy momentum tensor:

$$T^{\mu\nu} = T^{\mu\nu}{}_{\rm eq} + \Gamma^{\mu\nu}{}_{\rm (visc)}$$

so that the standard conservation equation implies

$$u_\mu T^{\mu\nu}{}_{{\rm eq};\nu} = -u_\mu T^{\mu\nu}{}_{({\rm visc});\nu} = \Theta \mathscr{S}^\mu{}_{;\mu}\,, \tag{4.9}$$

where

$$\mathscr{S}^\mu = \mathscr{S}u^\mu \tag{4.10}$$

is the entropy flux. A natural generalization of (4.9) in the scale-covariant framework would be

$$u_\mu T^{\mu\nu}{}_{{\rm eq}*\nu} = \Theta \mathscr{S}^\mu{}_{*\mu}\,. \tag{4.11}$$

Assuming $\Pi(\mathscr{S}^{\mu}) = \Pi_s$, and with $T^{\mu\nu}{}_{\rm eq}$ being given by (4.2), the above equation can be written as

$$\dot{\rho} + (\rho + p)\theta + [(1 - \Pi_g)\rho + 3p]\dot{\phi} = \Theta(\dot{\mathscr{S}} + \mathscr{S}\Theta + (\Pi_s + 4)\mathscr{S}\dot{\phi}),$$

where we have used (2.4b) to expand the right hand side. Again assuming homogeneity over the volume element V, the above can be written as

$$dU + pdV + [(1 - \Pi_g)\rho + 3p - (\Pi_s + 4)\mathscr{S}\Theta]Vd\phi = \Theta d(\mathscr{S}V) \equiv \Theta dS. \tag{4.12}$$

Thus we see that (4.12), and therefore (4.11), is incompatible with (4.8) unless $\Pi_s = -4$, which in turn requires S to be a dimensionless quantity. This is reasonable since S is a measure of the multiplicity of available states, from a statistical view point. In classical thermodynamics, the dimension of entropy is not determined *a priori*. In fact if $\Pi_s \neq -4$, the appearance of $\mathscr{S}$ and Θ as coefficient of $d\phi$ would make it impossible to interpret ϕ as simply an additional thermodynamic parameter.

It should be noted that ϕ possesses two properties which usually are not associated with thermodynamic parameters. (*a*) *Local universality*.—All systems in any spacetime neighborhood have the same externally imposed ϕ. (*b*) *Long scale of variation*.—In accordance with remarks made in § II*a* concerning the variation of β, variations of ϕ must have a cosmological scale.

b) *Integrability Conditions*

We next study some consequences of (4.8) when the equation of state can be written as

$$p = \Gamma\rho, \tag{4.13}$$

so that (4.8) can be written as

$$\Theta dS = dU + \Gamma\rho dV + \Omega U d\phi, \tag{4.14}$$

where

$$\Omega \equiv 3\Gamma + 1 - \Pi_g \tag{4.15}$$

and Γ is assumed to be a constant. If we take Θ, V, ϕ to be independent variables, (4.14) can also be written as

$$\Theta dS = V\frac{\partial\rho}{\partial\Theta}d\Theta + \left[(1 + \Gamma)\rho + V\frac{\partial\rho}{\partial V}\right]dV + \left(\frac{\partial\rho}{\partial\phi} + \Omega\rho\right)Vd\phi. \tag{4.14'}$$

The three integrability conditions for (4.14′) are

$$\Gamma\frac{\partial\rho}{\partial\Theta} = (1 + \Gamma)\frac{\rho}{\Theta} + \frac{V}{\Theta}\frac{\partial\rho}{\partial V}, \tag{4.16a}$$

$$\Omega\frac{\partial\rho}{\partial\Theta} = \frac{1}{\Theta}\left(\Omega\rho + \frac{\partial\rho}{\partial\phi}\right), \tag{4.16b}$$

$$\Omega\frac{\partial\rho}{\partial\phi} = \Omega\left(\rho + V\frac{\partial\rho}{\partial V}\right). \tag{4.16c}$$

Assuming separable variables, the above system of equation can be solved for ρ, which takes the form

$$\rho = \rho_0 V^{\alpha_1}\Theta^{\alpha_2}\beta^{\alpha_3}, \tag{4.17}$$

where ρ_0 is a constant and

$$\alpha_1 = \frac{\partial \ln \rho}{\partial \ln V}; \qquad \Gamma\alpha_2 = 1 + \Gamma + \alpha_1; \qquad \Gamma\alpha_3 = (1 + \alpha_1)\Omega. \tag{4.18}$$

We observe that if $dS = 0$, (4.14′) can easily be integrated to give

$$\rho V^{1+\Gamma}\beta^{\Omega} = \text{constant}. \tag{4.19a}$$

Making use of the expression (4.17), we find

$$\Theta V^{\Gamma}\beta^{\Omega} = \text{constant}, \tag{4.19b}$$

which we shall call the "adiabatic temperature scaling law." From equations (4.19), we obtain an equation independent of β:

$$\rho V/\Theta = \text{constant}. \tag{4.20}$$

More generally, if we substitute (4.17) into (4.14′), it can be shown, with a little algebra, that

$$dS = S_0'(\beta^{\Omega}\Theta V^{\Gamma})^{(1+\alpha_1)/\Gamma} d[\ln(\Theta\beta^{\Omega}V^{\Gamma})] .$$

Consequently we find the expression for the entropy:

$$S = S_0(\Theta V^{\Gamma}\beta^{\Omega})^{(1+\alpha_1)/\Gamma} + \text{const.} \quad \text{if } \alpha_1 \neq -1 , \tag{4.21a}$$

or

$$S = S_0 \ln(\Theta V^{\Gamma}\beta^{\Omega}) + \text{const.} \quad \text{if } \alpha_1 = -1 . \tag{4.21b}$$

Obviously (4.20) is merely a special case of (4.21).

c) *Classical Ideal Gas*

To better understand the physical contents of the thermodynamic relations derived above, we shall study two particular systems: a classical ideal gas in the remainder of § IV and radiation in § V.

First we note that the ρ appearing in previous equations of this section denotes the total energy density. We can obtain the internal energy density u by subtracting from ρ the rest mass density,

$$u = \rho - mn , \tag{4.22}$$

where m is the mass of the particles and n is the number density, satisfying the following equation (see Paper I):

$$\dot{n} + n\theta + (1 - \Pi_g)n\dot{\beta}/\beta = 0 . \tag{4.23}$$

It can be shown from (4.3), (4.22) and (4.23) that

$$\dot{u} + (u + p)\theta + [(1 - \Pi_g)u + 3p]d(\ln\beta)/dt = 0 , \tag{4.24}$$

which is formally identical to (4.3) with ρ replaced by u. Thus all the previously derived thermodynamical relations are also valid when ρ is replaced by u. (Notice that the validity of [4.24] is not restricted to nonrelativistic systems.)

To determine α_1 in (4.17), we first define the gas "kinetic temperature" T in the usual way:

$$p = nkT , \tag{4.25}$$

where k is the Boltzmann constant. Next, (4.23) can be integrated to give (see also Paper I, eq. [3.3])

$$nV\beta^{1-\Pi_g} = \text{const.} \tag{4.26}$$

or

$$n = \frac{N_0}{V}\beta^{\Pi_g - 1} = \frac{N_0}{V}\frac{1}{\beta G} .$$

Thus, we have

$$p = \frac{N_0}{V}kT\beta^{\Pi_g - 1} = \frac{N_0}{V}kT\frac{1}{\beta G} . \tag{4.27}$$

On the other hand, from (4.13) and (4.17),

$$p = \Gamma u_0 V^{\alpha_1}\Theta^{\alpha_2}\beta^{\alpha_3} \tag{4.28}$$

which, upon comparison with (4.27), yields

$$\alpha_1 = -1 . \tag{4.29}$$

This in turn gives

$$\alpha_2 = 1 , \qquad \alpha_3 = 0 . \tag{4.30}$$

Hence identifying the above expressions for p, we find the relation between Θ and T:

$$\Theta = c_1 T\beta^{\Pi_g - 1} , \qquad c_1 \equiv \frac{kN_0}{\Gamma u_0} . \tag{4.31}$$

Finally, we have an explicit form for (4.17):

$$u = u_0 V^{-1}\Theta = \Gamma^{-1}nkT . \tag{4.32}$$

Clearly the average kinetic energy per particle is

$$\langle \epsilon \rangle = \frac{u}{n} = \frac{1}{\Gamma} kT , \tag{4.33}$$

thus reconfirming that T is indeed the gas kinetic temperature.

Before proceeding to consider the thermodynamics of radiation, we shall comment on the results of this section. (1) The introduction of T can be carried through for ideal gas of arbitrary Γ. (2) The quantity α_1 can be obtained more directly by observing that, if β is constant, the entire formulation must reduce to that of standard theory. Since α_1 is given by a partial derivative with respect to V, it must be identical to that of the standard theory which, for an ideal gas, is -1. (3) Equations (4.25), (4.32), and (4.33) are identical to those of the standard theory, as indeed they should be. For the kinetic temperature having been defined by any one of the three relations, the other two follow trivially from the kinetic picture of classical ideas gas. We have merely demonstrated that our modified thermodynamics can be compatible with elementary kinetic theory. Furthermore, it would be wrong to conclude that the effect of β is not observable thermodynamically. In fact, from (4.21b) we have the entropy

$$S = S_0 \ln (TV^\Gamma \beta^{3\Gamma}) + \text{constant} , \tag{4.34}$$

and consequently the adiabatic scaling law for a classical ideas gas is

$$TV^\Gamma \beta^{3\Gamma} = \text{constant} . \tag{4.35}$$

Thus with an imposed β variation, the product TV^Γ must adjust and vary if the system is adiabatically contained.

V. RADIATION

a) *Thermodynamic Relations*

Radiation is considered here to be an aggregate of massless particles moving along null, in-geodesics governed by equation (3.10). Since $k^\mu k_\mu = 0$, the energy-momentum tensor for radiation is therefore traceless. Imposing this condition on (4.2), we find immediately the equation of state:

$$p = \tfrac{1}{3}\rho_\gamma ; \qquad \Gamma = \tfrac{1}{3} . \tag{5.1}$$

All the thermodynamic relations for this system can be explicitly given if we further specify α_1. We have learned from the last section that the simplest method is to consider β constant so that the present formulation reduces to standard theory. Then Kirchhoff's law states that $\rho_{\gamma\nu} = \rho_{\gamma\nu}(\Theta)$, independent of volume ($\rho_{\gamma\nu}$ is the equilibrium spectral distribution for radiation). Consequently the radiation energy density

$$\rho_\gamma = \int d\nu \rho_{\gamma\nu} \tag{5.2}$$

is independent of V, and hence

$$\alpha_1 = \frac{\partial \ln \rho_\gamma}{\partial \ln V} = 0 . \tag{5.3}$$

Equations (4.15) and (4.18) then yield

$$\Omega = 2 - \Pi_g , \qquad \alpha_2 = 4 , \qquad \alpha_3 = 3(2 - \Pi_g) , \tag{5.4}$$

which, along with (4.17), gives

$$\rho_\gamma = \rho_0' \beta^{3(2-\Pi_g)} \Theta^4 \tag{5.5a}$$

or

$$\rho_\gamma = a\beta^{\Pi_g+2} T^4 = aT^4(\beta^2/G) , \tag{5.5b}$$

where the gas kinetic temperature is used in equation (5.5b). It should be pointed out that while classical thermodynamics precludes the dependence of $\rho_{\gamma\nu}$ on any thermodynamic parameter other than Θ, due to the universality of ϕ (or β), the latter cannot be excluded and hence one must assume

$$\rho_{\gamma\nu} = \rho_{\gamma\nu}(\Theta, \beta) . \tag{5.6}$$

Thus the appearance of β in (5.5) is a necessary consequence of our thermodynamic postulates.

The radiation entropy is obtained from (4.21a):

$$\begin{aligned} S &= S_0(\Theta V^{1/3}\beta^{2-\Pi_g})^3 + \text{const.} \\ &= S_0(TV^{1/3}\beta)^3 + \text{const.} \end{aligned} \tag{5.7}$$

Hence the adiabatic scaling law is

$$TV^{1/3}\beta = \text{constant}. \tag{5.8a}$$

It is sometimes convenient to replace $V^{1/3}$ by R, the linear dimension of the system, so that

$$TR\beta = \text{constant}. \tag{5.8b}$$

Note the elimination of T from (5.5b) and (5.8b) gives

$$\rho_\gamma R^4 \beta^{2-\Pi_g} = \text{constant}, \tag{5.9}$$

which is a relation already derived in Paper I. It can be obtained directly from (4.3), assuming the equation of state (5.1).

These results, along with those of the previous section, allow us to study radiation and matter in equilibrium. We can consider the internal energy to be

$$\rho = \rho_\gamma + u \tag{5.10}$$

and the total pressure

$$P = p_\gamma + p = \tfrac{1}{3}\rho_\gamma + \Gamma u. \tag{5.11}$$

If we substitute these expressions into (4.24) and use (5.5b) and (4.33) for ρ_γ and u, we find, for adiabatic processes, the exact relation

$$V^{\sigma+1}\beta^{3(\sigma+1)}T^{3\sigma+1/\Gamma} = \text{constant}, \tag{5.12}$$

where

$$\sigma \equiv \tfrac{4}{3}\rho_{\gamma 0}T^3 V\beta^3/N_0 k. \tag{5.13}$$

Clearly, if $\sigma \gg 1$, (5.12) reduces to radiative scaling (5.8a). And if $\sigma \ll 1$, (5.12) reduces to (4.35). We note furthermore that σ is constant in the limit $\sigma \gg 1$.

b) *Modified Wien's Displacement Law*

Equations (5.5) and (5.9) give us only information about the radiative energy density, integrated over frequency. As is well known, a complete determination of the equilibrium spectral distribution can emerge only from quantum statistics. Nevertheless, classical thermodynamics already provides some information about the distribution function, as is given by Wien's displacement law. We shall see how the latter is modified in the present framework.

If radiation having equilibrium distribution is prepared within perfectly reflecting walls, thermodynamic arguments lead to the conclusion that if the volume of the enclosure is adiabatically varied, the radiation would retain an equilibrium distribution, having a shifted temperature so as to adjust to the total amount of energy within the enclosure. This conclusion remains valid in the present context; therefore, without repeating well known derivations (see, e.g., Born 1964), we can write

$$\frac{\partial}{\partial V}(\rho_{\gamma\nu}V) = \tfrac{1}{3}\nu\frac{\partial\rho_{\gamma\nu}}{\partial\nu} \tag{5.14}$$

with the following remarks: (1) The quantity $\rho_{\gamma\nu}$ is now considered a function of ν, V, and ϕ. We are now considering adiabatic processes for which (5.8a) holds; hence V can replace the temperature as an independent variable. (2) The partial derivatives in (5.14) are taken with ϕ held fixed. Thus the classical derivation of (5.14) is indeed valid, for, as can be seen from (3.12), the Doppler shift relation reduces to standard form.

If we write

$$\rho_{\gamma\nu} = \rho(\beta)\rho_\nu(V), \tag{5.15}$$

equation (5.14) gives immediately

$$\rho_\nu \sim \nu^3 F(\nu^3/V), \tag{5.16a}$$

which can also be written, using (5.8a),

$$\rho_\nu \sim \nu^3 f(\nu/\beta T). \tag{5.16b}$$

Here f is an arbitrary function of the argument.

Note that the frequency integrated distribution must agree with (5.5); hence combining (5.15) and (5.16b) and integrating, we find

$$\rho(\beta) \sim \beta^{\Pi_g-2},$$

and hence

$$\rho_{\gamma\nu} = \beta^{\Pi_g - 2}\nu^3 f(\nu/\beta T) = \frac{1}{\beta^2 G}\nu^3 f(\nu/\beta T) . \tag{5.17}$$

It follows obviously that, if a maximum exists for $\rho_{\gamma\nu}$, $\nu_{\max}$ satisfies the modified Wien's displacement law,

$$\nu_{\max} = \text{constant}\cdot\beta T . \tag{5.18}$$

c) *Radiation Kinetic Temperature*

Although (5.17) does not at all specify the equilibrium distribution, some useful information can still be extracted from its functional form. Thus, with (3.15), we find photon number density per frequency interval:

$$n_\nu = \frac{\rho_{\gamma\nu}}{\epsilon_\nu} = \beta^{\Pi_g - 1}\nu^2 f(\nu/\beta T) . \tag{5.19}$$

The integrated photon number density can easily be obtained:

$$n_\gamma = \int d\nu n_\nu \sim \beta^{\Pi_g + 2} T^3 , \tag{5.20}$$

so we find the average photon energy as (using 5.5b)

$$\langle \epsilon_\gamma \rangle = \frac{\rho_\gamma}{n_\gamma} \sim T . \tag{5.21}$$

Thus, the gas kinetic temperature, which is a measure of the average particle energy, is also a good measure of the average radiation energy.

d) *The* 3 K *Background Radiation*

The observed 3 K background radiation has long been considered to be a remnant of equilibrium radiation from an earlier epoch. We briefly review the well known chain of reasoning leading to the above conclusion, so as to facilitate subsequent discussion in the scale covariant framework:

i) As the universe expands, individual photons suffer a frequency shift given by

$$\nu/\nu_0 = R_0/R . \tag{5.22}$$

ii) Standard conservation law gives for adiabatically expanding radiation

$$\rho_\gamma R^4 = \text{constant} , \tag{5.23}$$

as can be seen by setting $\beta = 1$ in (5.9). If the radiation consists of noninteracting photons, the above conservation law holds for each spectral interval:

$$d\nu \rho_{\gamma\nu} R^4 = \text{constant} . \tag{5.24}$$

iii) For radiation in equilibrium, the spectral distribution is given by

$$\rho_{\gamma\nu} \sim \nu^3[\exp(h\nu/kT) - 1]^{-1} \equiv \nu^3 f(\nu/T) . \tag{5.25}$$

If the radiation subsequently becomes a gas of noninteracting photons streaming freely in an expanding universe, the spectral distribution at a later stage would be, according to (5.22) and (5.24),

$$\rho_{\gamma\nu_0} d\nu_0 = d\nu_0 \nu_0^3 f(\nu_0/T_0) , \tag{5.26}$$

where

$$T_0 = T(R/R_0) \tag{5.27}$$

is a scaled temperature inferred from an observed distribution and does not have any thermodynamic significance.

iv) Mean free path estimates suggest that the observed radiation is not in thermal equilibrium with matter at present, whereas in the past an equilibrium state is expected to have existed due both to ionization and to higher matter density. It is therefore compelling to regard the observed radiation with a spectrum given by (5.26) to be a remnant of the past.

We note that, given the initial equilibrium distribution and the redshift relation (5.22), the scaled radiation would keep the equilibrium form (5.26) if and only if the conservation law (5.24) holds. For this reason, people believe that any theory which alters (5.24) is unable to explain the observed microwave radiation in the above natural way as a remnant of equilibrium radiation. In fact, many people attach such a great significance to this explanation of the background radiation that a failure to do so had prompted the rejection of theories which modify (5.24) (Hönl and Dehnen 1968; Jordan 1968; Steigman 1978). We shall see in the following that such worries are unfounded.

In the scale-covariant framework, (5.22) is unaltered as was shown by (3.14). Due to our modified conservation law however, (5.24) must be changed to

$$\beta^{2-\Pi_g} R^4 \rho_{\gamma\nu} d\nu = \text{constant}. \tag{5.28}$$

Just as we did not need the more explicit form of (5.25) to discuss the scaling of equilibrium distribution in the standard theory, we shall start with the modified equilibrium distribution given by (5.17). Using (5.22) and (5.28), it can then be shown that a freely streaming photon gas having (5.17) as the initial distribution would evolve to have the following spectrum:

$$\rho_{\nu_0} d\nu_0 = \beta_0^{\Pi_g - 2} \nu_0^3 f\left(\frac{\nu_0}{\beta_0 T_0}\right) d\nu_0 , \tag{5.29}$$

where

$$T_0 = T \frac{\beta R}{\beta_0 R_0} \tag{5.30}$$

is again a scaled temperature having no thermodynamic significance. We note that, analogous to (5.26), (5.29) also has the appearance of an equilibrium distribution. Thus *in both standard theory and the scale covariant theory, freely streaming radiation initially having an equilibrium distribution would retain the equilibrium form with a scaled temperature.*

At this point we add a few words concerning the terminology: blackbody radiation. As is well known, the spectrum from a blackbody is identical to the equilibrium radiation spectrum. In the standard theory, the latter is given by (5.25). In the present theory, the equilibrium distribution takes the more general form (5.17) which has, compared to (5.25) an extra multiplicative factor, $\beta^{\Pi_g - 2}$. We shall continue to call (5.25) the blackbody spectrum, and we call the multiplication factor of (5.17) the "grayness" factor. At any given time t, (5.25) and (5.17) are different by a constant $\beta^{\Pi_g - 2}(t)$ (independent of ν) which can be reduced to unity by a normalization process applied to the function β at that particular time. Indeed, we have been setting $\beta_0 = 1$ for the present epoch so that the present theory implies, according to (5.29), that the remnant of equilibrium radiation would appear black as observed today. Thus, at any one particular time there is no observable difference between (5.17) and (5.25). This brings us back to a point made several times before: it is the derivative of β, not the absolute value of β, that is observable.

The above interpretation of the observed blackbody radiation as a remnant of equilibrium radiation of the past is applicable for arbitrary gauges, i.e., Π_g is arbitrary. In particular, if $\Pi_g = 2$, the grayness factor reduces to unity identically. This way of reproducing the blackbody spectrum has been discussed previously (Canuto and Hsieh 1978).

VI. SUMMARY

A mathematical language for the description of astrophysical phenomena, consistent with the scale-covariant theory of gravitation, has been established. To give an overview of its structure, we gather the main results below.

a) *Description of Individual Particles*

From geometrical considerations, we have the in-geodesic equation

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\nu\rho} \frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} + \frac{dx^\mu}{d\lambda}\frac{dx^\rho}{d\lambda}(\ln \beta)_\rho - (\ln \beta)^\mu \left(\frac{dx^\rho}{d\lambda}\frac{dx_\rho}{d\lambda}\right) = 0 , \tag{6.1}$$

where $dx^\mu/d\lambda$ is the tangent vector of the in-geodesic path. The equation applies to paths of both massive and massless particles. For the former, we write

$$\frac{dx^\mu}{d\lambda} \equiv \frac{dx^\mu}{ds} \equiv v^\mu \quad \text{with} \quad v^\mu v_\mu = 1 . \tag{6.2a}$$

For the latter,

$$\frac{dx^\mu}{d\lambda} \equiv k^\mu \quad \text{with} \quad k^\mu k_\mu = 0 . \tag{6.2b}$$

The 4-momenta of particles are defined in terms of the geometrical parameters

$$p^\mu = mv^\mu \quad \text{or} \quad p^\mu = hk^\mu . \tag{6.3}$$

An observer with a 4-velocity u^μ will find the energies of the particles to be

$$\epsilon = u_\mu p^\mu , \tag{6.4}$$

and the observed photon frequency ν is given by

$$\nu = \beta u_\mu k^\mu \tag{6.5}$$

so that

$$\epsilon_\nu = \frac{h}{\beta}\nu . \tag{6.6}$$

b) Hydro-Thermodynamics

The energy momentum tensor is co-covariantly conserved:

$$T^{\mu\nu}{}_{*\nu} = 0 . \tag{6.7}$$

The particle number density satisfies a modified conservation equation:

$$(nu^\mu)_{;\mu} + (1 - \Pi_g)(\ln \beta)_\mu nu^\mu = 0 \tag{6.8a}$$

or

$$n = \frac{N_0}{V}\cdot\frac{1}{G\beta} , \tag{6.8b}$$

where in (6.8b) we have displayed the dependence on G explicitly. It should be noted, however, that by definition

$$G = \beta^{-\Pi_g}\bar{G} , \tag{6.9}$$

where $\bar{G}$, being the gravitational constant in Einstein units, is a true constant. Thus the function form of G cannot be arbitrarily specified independent of β.

Because of the modified conservation laws, β must enter as a parameter into the thermodynamic relations in addition to the usual volume and temperature. For a system with the equation of state

$$p = \Gamma\rho , \tag{6.10}$$

we find

$$\rho = \rho_0 V^{\alpha_1} T^{(1+\Gamma+\alpha_1)/\Gamma} \beta^{2+3\alpha_1}/G , \tag{6.11}$$

where

$$\alpha_1 = \frac{\partial \ln \rho}{\partial \ln V} . \tag{6.12}$$

The adiabatic scaling law also involves the function β:

$$TV^\Gamma \beta^{3\Gamma} = \text{constant} . \tag{6.13}$$

For classical ideal gas with $\alpha_1 = -1$, the internal energy u can be written as

$$u = \frac{3}{2}\frac{N_0}{V} kT \frac{1}{G\beta} . \tag{6.14a}$$

The adiabatic scaling law is

$$TV^{2/3}\beta^2 = \text{constant} . \tag{6.15a}$$

Furthermore, the rest mass density for the gas can be obtained from (6.8b):

$$\rho_m = m\frac{N_0}{V}\frac{1}{G\beta} . \tag{6.16}$$

For a system of radiation, $\Gamma = \frac{1}{3}$, $\alpha_1 = 0$. The energy density has the expression

$$\rho_\gamma = \rho_{\gamma 0} T^4 \frac{\beta^2}{G} = \rho_{\gamma 0} V^{-4/3} \frac{1}{\beta^2 G}, \tag{6.14b}$$

where in the second equality, the adiabatic scaling

$$TV^{1/3}\beta = \text{constant} \tag{6.15b}$$

has been used. Furthermore, it has been shown that the spectral distribution of equilibrium radiation can be expressed as

$$\rho_{\gamma\nu} \sim (\beta^2 G)^{-1} \nu^3 f\left(\frac{\nu}{\beta T}\right), \tag{6.17}$$

where f is an arbitrary function of its argument.

c) *Kinetic Interpretation*

Combining (*a*) and (*b*), we have an elementary kinetic interpretation of the thermodynamic results. For example, (6.8b) and (6.14a) give the average kinetic energy of a particle in the gas

$$\langle\epsilon\rangle = \tfrac{3}{2} kT. \tag{6.18a}$$

Equation (6.6) along with (6.17) yields the photon number density in equilibrium radiation:

$$n_\gamma = N_{\gamma 0} T^3 \frac{\beta^2}{G} = \frac{N_{\gamma 0}}{V} \frac{1}{\beta G}, \tag{6.19}$$

which in turn gives, with (6.14b), the average energy per photon,

$$\langle\epsilon_\gamma\rangle \sim kT. \tag{6.18b}$$

We note that in the second relation of (6.19), the scaling law (6.15b) has been used. The similarity of (6.19) and (6.8b) should be noted: In standard relativity, the total photon number of equilibrium radiation is adiabatically conserved. In the present framework, the adiabatic conservation law is modified in exactly the same manner as the particle number conservation law.

Given the above results, we can return to the theme in the Introduction with a clearer understanding: internal consistency demands many modifications in astrophysical calculations pertaining to the study of the effects of a varying gravitational constant. We have already discussed the example of the spectrum of equilibrium radiation. In a similar manner, the radiative energy density cannot be simply written as aT^4. The factor β^2/G in (6.14b) must be included if one insists that T represents the average kinetic energy of the gas particles. This in turn forces modification of the radiative transfer equation which enters into stellar evolution calculations. Work on this problem and on other cosmological consequences will be reported in subsequent papers.

d) *Hawking's (1975) T versus GM Relation*

The modified energy versus frequency relation, (2.15), has a significant implication. In fact, from (6.17) we can derive that the maximum of the $\rho_{\gamma\nu}$ versus T function occurs at

$$\frac{h\nu}{\beta kT} \sim \text{const.} \tag{6.20}$$

or

$$T \sim \frac{hc}{\lambda\beta k}. \tag{6.21}$$

Now, if an object of mass M emits radiation, the maximum wavelength will be proportional to $R_E = G_E M_E/c^2$. We therefore have

$$\lambda \sim \frac{1}{\beta} R_E = \frac{G_E M_E}{\beta c^2} = \frac{GM}{c^2}, \tag{6.22}$$

since we have already shown that $GM\beta = G_E M_E = \text{const.}$ Substituting (6.22) into (6.21), we finally obtain

$$T \sim \frac{hc^3}{GM\beta k} = \frac{hc^3}{G_E M_E k}, \tag{6.23}$$

which is Hawking's relation.

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